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Factorisation of Operators II.

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Abstract

We extend the methods of a previous paper[1], factorising the general, scalar, third order differential operator, and obtain a Miura transformation for the Boussinesq equation. We give a general method for deriving a recursion operator and apply this method to the factorised eigenvalue problem. We also give a Hamiltonian structure associated with the factorised eigenvalue problem. We derive several isospectral flows, some of Klein-Gordon type.

1. INTRODUCTION.

In a recent paper [1] (referred to below as I) we discussed the factorisation of some particular third order Lax operators, with emphasis on 'Miura transformations' and their associated modified equation. In this paper we consider the general third order scattering operator

$$L = \partial^3 + v\partial + \tfrac{1}{2}v_x + w \tag{1.1}$$

(where $\partial \equiv \partial_x \equiv \frac{\partial}{\partial x}$) of which the two operators discussed in I are special cases. This is the scattering operator for the well known Boussinesq equation [2,3].

In section 2 we factorise this operator, deriving an associated 3 x 3 matrix scattering problem. The matrices involved all belong to the Lie algebra $sl(3,\mathbb{R})$, a point which we develop in later sections. We derive the first two polynomial isospectral flows, the first of which is trivial, being just the translation equation, and the second of which is related to the Boussinesq equation by a Miura transformation[4]. We shall refer to this system as the 'modified Boussinesq equation'.

Polynomial flows of higher order may be constructed recursively; in sections 3 and 4 we consider a systematic method of constructing the recursion operator. In section 3 we consider the matrix scattering problem in the adjoint and coadjoint representations of $sl(3,\mathbb{R})$. The coadjoint equations are satisfied by analogues of the usual 'squared eigenfunctions'[5]. By eliminating variables we derive an integro-differential operator, whose adjoint is shown to be a recursion operator [6] relating isospectral flows. In section 4 we give a different derivation of the recursion operator and exploit the relationship between the two expressions to derive a Hamiltonian structure. We indicate how the recursion operator generates the isospectral flows discussed in section 2. We then calculate, as a special case of the fifth order system, the 'modified Kupershmidt equation' discussed in I.

In section 5 we consider isospectral flows of Klein-Gordon type. These systems, generalisations of the sinh-Gordon equations, are related to the three-particle, periodic Toda lattice. We generalise this further to the n-particle lattice in another paper[7].

2. FACTORISATION OF THE SCATTERING OPERATOR.

The Boussinesq equation

$$V_{tt} = -\frac{1}{3}\left(V_{4x} + 2(V^2)_{xx}\right) \tag{2.1}$$

written as the system

$$\begin{aligned} V_t &= 2W_x \\ W_t &= -\frac{1}{6}\left(V_{3x} + 4VV_x\right) \end{aligned} \tag{2.2}$$

may be represented as a Lax pair[2,3] :

$$L_t = [P\ L] \tag{2.3}$$

where

$$\begin{aligned} L &= \partial^3 + V\partial + \frac{1}{2}V_x + W \\ P &= \partial^2 + \frac{2}{3}V \end{aligned} \tag{2.4}$$

The scattering operator L may be decomposed into the product

$$L' = (\partial + y - z)(\partial + 2z)(\partial - y - z) \tag{2.5}$$

provided we can find y and z such that

$$\begin{aligned} V &= -(2y_x + 3z^2 + y^2) \\ W &= -(z_{xx} + 3yz_x + zy_x + 2z(y^2 - z^2)) \end{aligned} \tag{2.6}$$

The eigenvalue problem of L :

$$L\Psi_1 = \zeta^3 \Psi_1 \tag{2.7}$$

may then be written as

$$\frac{\partial}{\partial x}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} = \begin{pmatrix} y+z & \zeta & 0\\ 0 & -2z & \zeta \\ \zeta & 0 & -y+z\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} \tag{2.8}$$

The matrix on the right of (2.8) is an element of the Lie algebra $sl(3,\mathbb{R})$ (traceless, 3 x 3 matrices); it can be expanded in terms of any convenient basis of that algebra. Such a basis, together with its commutation relations, is given in the appendix.

With respect to this basis, equation(2.8) is written

$$\partial \underset{\sim}{\Psi} = (\zeta R_1 + y R_2 + z R_5)\underset{\sim}{\Psi} \tag{2.9}$$

where $\underset{\sim}{\Psi} = (\Psi_1, \Psi_2, \Psi_3)^T$.

The general eigenvalue problem associated with $sl(3,\mathbb{R})$ (keeping ζ in the same position) is

$$\partial \underset{\sim}{\Psi} = u \underset{\sim}{\Psi} \tag{2.10}$$

where $u = \sum_{i=1}^{8} u^i R_i$, $u^1 \equiv \zeta$ and the remaining u^i are arbitrary functions.

Consider a time evolution of $\underset{\sim}{\Psi}$ of the form

$$\partial_t \underset{\sim}{\Psi} = A \underset{\sim}{\Psi} \tag{2.11}$$

where $A = \sum_{i=1}^{8} A^i R_i \in sl(3,\mathbb{R})$ and the A^i are functions of u^k, their x-derivatives, and ζ.

If (2.11) is to be consistent with (2.10), then

$$\partial_t u - \partial A + [u\ A] = 0 \tag{2.12}$$

In terms of the basis $\{R_i\}$ equation (2.12) becomes

$$\partial_t u^k - \partial A^k + u^i A^j C^k_{ij} = 0 \tag{2.13}$$

where C^k_{ij} are the structure constants of $sl(3,\mathbb{R})$ with the given basis.

When the eigenvalue problem is associated with the factorisation of a scalar operator, taking the form of (2.9), equations (2.13) become (with $u^2 = y$ and $u^5 = z$),

$$\begin{aligned}
\partial A^1 &= 2(u^5 A^6 - u^2 A^3)\\
\partial A^2 &= \partial_t u^2 - 3\zeta A^7\\
\partial A^3 &= \zeta A^2 - u^2(A^1 + A^3) - u^5 A^6\\
\partial A^4 &= 3\zeta A^3 + 3u^5(A^7 - A^8) - u^2 A^4\\
\partial A^5 &= \partial_t u^5 + \zeta A^4\\
\partial A^6 &= 3u^5(A^1 - A^3) - 3\zeta A^5 + u^2 A^6\\
\partial A^7 &= -\zeta A^6 + u^2(A^7 + A^8) + u^5 A^4\\
\partial A^8 &= 2(u^2 A^7 - u^5 A^4)
\end{aligned} \tag{2.14}$$

A trivial solution of (2.14) is

$$A = u \tag{2.15}$$

corresponding to the translation flow

$$u_t = u_x \tag{2.16}$$

where $u = (u^1, u^5)^T$.

A more interesting solution is given by

$$
\begin{aligned}
A^1 &= A^4 = A^7 = 0 \\
A^2 &= 2\,u^2 u^5 + u^5_{\ x} \\
A^3 &= \zeta\, u^5 \\
A^5 &= \tfrac{1}{3}\left((u^2)^2 - 3(u^5)^2\right) - \tfrac{1}{2}\, u^2_{\ x} \\
A^6 &= \zeta\, u^2 \\
A^8 &= \zeta^2
\end{aligned}
\tag{2.17}
$$

which generates the flow:

$$
\begin{aligned}
u^2_{\ t} &= u^5_{\ xx} + 2\left(u^2 u^5\right)_x \\
u^5_{\ t} &= -\tfrac{1}{3}\, u^2_{\ xx} - \tfrac{1}{3}\left(3(u^5)^2 - (u^2)^2\right)_x
\end{aligned}
\tag{2.18}
$$

which we refer to as the modified Boussinesq equation, it being related to (2.2) by the 'Miura transformation' (2.6).

These flows and a sequence of higher order polynomial flows may be generated recursively, as will be discussed in the following sections.

3. CO-ADJOINT STRUCTURE AND SQUARED EIGENFUNCTIONS.

The integrability condition (2.12):

$$A_x - [u\,A] = u_t \tag{3.1}$$

is considered as a linear, inhomogeneous equation for A. The corresponding homogeneous equation

$$\Psi_x = [u\;\Psi] \tag{3.2}$$

is the adjoint representation of (2.10), Ψ being an element of $s\ell(3,\mathbb{R})$. Equation (3.2) may be derived from the requirement that the integral

$$\int \langle \Phi\,,\ \Psi_x - [u\;\Psi] \rangle\, dx \tag{3.3}$$

be stationary with respect to variations in Φ, where Φ is an element of the vector space dual to $s\ell(3,\mathbb{R})$ and $\langle\ ,\ \rangle$ denotes the inner product between elements of these spaces.

Requiring this same integral to be stationary with respect to variations in Ψ we obtain the adjoint † of (3.2):

$$\Phi_x = -\{u\,,\ \Phi\} \tag{3.4}$$

where $\{u, \Phi\} = ad^*_u\,\Phi$ is defined by

$$\langle \{u,\Phi\}, \Psi \rangle = \langle \Phi\,,\ [u\;\Psi] \rangle \tag{3.5}$$

The linear transformation ad^*_u on the dual of the Lie algebra is a representation, called the co-adjoint [8], of the element (-u) of the algebra.

† It is unfortunate that the word 'adjoint' has several distinct meanings.

If Ψ is expanded in terms of the basis $\{R_i\}$ and Φ in terms of its dual basis, then

$$\langle \Phi , \Psi \rangle = \sum_{i=1}^{8} \Phi_i \Psi^i \tag{3.6}$$

so that (3.4) becomes

$$\Phi_{k,x} = - u^j C_{jk}^{i} \Phi_i \tag{3.7}$$

Equations (3.1) and (3.4) imply

$$\int \langle \Phi , u_t \rangle \, dx = 0 \tag{3.8}$$

which should be compared with (3.13) below. This equation relates the time evolutions of the eigenvalues and the field variables. In the case developed below these are respectively u^1 and the pair $\{u^2, u^5\}$.

Squared Eigenfunctions

Consider the integral

$$\int \phi (\psi_x - u\psi) \, dx \tag{3.9}$$

where ϕ and ψ are row and column three-vectors and $u \in sl(3,\mathbb{R})$, and require that it be stationary with respect to variations of ϕ and ψ, obtaining

$$\psi_x = u \psi \tag{3.10},$$

which is just (2.10); and its adjoint

$$\phi_x = - \phi u \tag{3.11}.$$

The matrix

$$\Psi = \psi \phi \tag{3.12}$$

is then easily seen to satisfy equation (3.2). If we now look for those deformations of u which leave the integral (3.9) stationary under the above mentioned variations, we obtain the condition

$$\sum_{i=1}^{8} \int \dot{u}^i \phi R_i \psi \, dx = 0 \tag{3.13}$$

where $u = \sum_{i=1}^{8} u^i R_i$ and $\dot{u}$ is the deformation of u.

It is easily seen that the functions

$$\begin{aligned}\Phi_i &= \phi R_i \psi \\ &= Tr(R_i \Psi) = Tr(R_i R_j) \Psi^j\end{aligned} \tag{3.14}$$

satisfy equation (3.7). The matrix $Tr(R_i R_j)$, known as the trace form, is important in the study of the Hamiltonian structure associated with the eigenvalue problem (2.10), as will be discussed in the next section.

These Φ_i and Ψ^i are the sets of squared eigenfunctions used for the theory of perturbed solitons [9]. They are also of importance in the theory of periodic solutions [10,11]. The system (3.7) then represents the squared eigenfunction equation, and gives us a way of constructing this explicitly, as will be seen below.

The Recursion Operator

We want to find an operator M^+ which maps one isospectral flow into another of higher order. To do this we consider the condition (3.13) and require that the eigenvalue $\zeta \equiv u^1$ be constant (for the eigenvalue problem(2.8)):

$$\int (\dot{u}^2 \Phi_2 + \dot{u}^5 \Phi_5) \, dx = 0 \tag{3.15}.$$

We are thus interested in a linear operator whose eigenfunction is the column vector $(\Phi_2, \Phi_5)^T$. We achieve this by eliminating the remaining components Φ_k from the system of equations (3.7):

$$
\begin{aligned}
\Phi_{1,x} &= u^2\,\Phi_3 - 3u^5\,\Phi_6\\
\Phi_{2,x} &= -\zeta\,\Phi_3\\
\Phi_{3,x} &= 2u^2\,\Phi_1 + u^2\,\Phi_3 - 3\zeta\,\Phi_4 + 3u^5\,\Phi_6\\
\Phi_{4,x} &= u^2\,\Phi_4 - \zeta\,\Phi_5 - u^5\,\Phi_7 + 2u^5\,\Phi_8\\
\Phi_{5,x} &= 3\zeta\,\Phi_6\\
\Phi_{6,x} &= -2u^5\,\Phi_1 + u^5\,\Phi_5 - u^2\,\Phi_6 + \zeta\,\Phi_7\\
\Phi_{7,x} &= 3\zeta\,\Phi_2 - 3u^5\,\Phi_4 - u^2\,\Phi_7 - 2u^2\,\Phi_8\\
\Phi_{8,x} &= 3u^5\,\Phi_4 - u^2\,\Phi_7
\end{aligned}
\tag{3.16}
$$

From this system of equations we derive:

$$
\begin{pmatrix}\Phi_6\\ \Phi_3\\ \Phi_1\end{pmatrix} = \frac{1}{\zeta}\,M_1\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix}
\qquad
\begin{pmatrix}\Phi_4\\ \Phi_7\\ \Phi_8\end{pmatrix} = \frac{1}{\zeta}\,M_2\begin{pmatrix}\Phi_6\\ \Phi_3\\ \Phi_1\end{pmatrix}
\tag{3.17}
$$

and

$$
\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix} = \frac{1}{\zeta}\,M_3\begin{pmatrix}\Phi_4\\ \Phi_7\\ \Phi_8\end{pmatrix}
$$

where

$$
M_1 = \begin{pmatrix} 0 & \frac{1}{3}\partial\\ -\partial & 0\\ -\partial^{-1}u^2\partial & -\partial^{-1}u^5\partial \end{pmatrix}
$$

$$
M_2 = \begin{pmatrix} u^5 & \frac{1}{3}(u^2-\partial) & \frac{2}{3}u^2\\ \partial+u^2 & -u^5 & 2u^5\\ \partial^{-1}\left(3(u^5)^2-(u^2)^2-u^2\partial\right) & \partial^{-1}\left(-u^5\partial+2u^2u^5\right) & 0 \end{pmatrix}
$$

and

$$
M_3 = \begin{pmatrix} u^5 & \frac{1}{3}(\partial+u^2) & \frac{2}{3}u^2\\ -\partial+u^2 & -u^5 & 2u^5 \end{pmatrix}
\tag{3.18}
$$

Thus, $(\Phi_2, \Phi_5)^T$ satisfies the equation:

$$
M\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix} = \zeta^3\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix}
\tag{3.19}
$$

where $M = M_3 M_2 M_1$.

Multiplying the integral (3.13) by ζ^3 and using (3.19) we get

$$
0 = \sum_{i=2,5}\int \dot{u}^i\,(M\,\Phi_i)\,dx = \sum_{i=2,5}\int (M^+\,\dot{u}^i)\,\Phi_i\,dx
\tag{3.20}
$$

so that $M^+\dot{u}$ is also an isospectral flow; hence M^+ is a recursion operator for the eigenvalue problem (2.8).

4. HAMILTONIAN STRUCTURE

In this section we find a relationship between the operator M and its adjoint, and exploit this to construct a Poisson bracket for the systems under discussion.

Since for simple Lie algebras, the trace form

$$g_{ij} = \mathrm{Tr}\,(R_i R_j) \tag{4.1}$$

is nonsingular, equation (3.14) implies

$$\Psi = g^{-1}\,\Phi \tag{4.2}$$

In particular

$$\begin{pmatrix}\Psi^2\\ \Psi^5\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{6}\end{pmatrix}\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix} = \hat{g}^{-1}\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix} \tag{4.3}$$

where $\hat{g}$ is the appropriate projection of g.

Hence, from equation (3.19) we get

$$\tilde{M}\begin{pmatrix}\Psi^2\\ \Psi^5\end{pmatrix} = \zeta^3\begin{pmatrix}\Psi^2\\ \Psi^5\end{pmatrix} \tag{4.4}$$

where $\tilde{M} = \hat{g}^{-1} M \hat{g}$.

We now look for A^k, polynomials in ζ, satisfying equation (3.1). Since $u^k{}_t$ must be independent of ζ, it is clear from comparison of (3.1) and (3.2) that A^2 and A^5 are polynomial in ζ^3 and their coefficients satisfy

$$\begin{pmatrix}A^2\\ A^5\end{pmatrix}^{(n)} = \tilde{M}\begin{pmatrix}A^2\\ A^5\end{pmatrix}^{(n+3)} \tag{4.5}$$

Since

$$\begin{pmatrix}u^2\\ u^5\end{pmatrix}_t = \partial\begin{pmatrix}A^2\\ A^5\end{pmatrix}^{(0)} \tag{4.6}$$

we find that

$$\partial\,\tilde{M}\,\partial^{-1} = \partial\,\hat{g}^{-1} M\,\hat{g}\,\partial^{-1} \tag{4.7}$$

is a recursion operator, relating isospectral flows of (2.9). It may be verified that this operator is equal to M^{+}.

Now we consider the adjoint of equation (3.19):

$$M^{+}\begin{pmatrix}\bar{\Phi}^2\\ \bar{\Phi}^5\end{pmatrix} = \zeta^3\begin{pmatrix}\bar{\Phi}^2\\ \bar{\Phi}^5\end{pmatrix} \tag{4.8}$$

where $\bar{\Phi}^2$ and $\bar{\Phi}^5$ are the 'adjoint wavefunctions'. However, since

$$M^{+} = \partial\,\hat{g}^{-1} M\,\hat{g}\,\partial^{-1} \tag{4.9}$$

by taking

$$\begin{pmatrix}\bar{\Phi}^2\\ \bar{\Phi}^5\end{pmatrix} = \partial\,\hat{g}^{-1}\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix} \tag{4.10}$$

we may solve equation (4.8). Thus, using the fact that eigenfunctions $\Phi(\zeta)$ at one eigenvalue are orthogonal to the adjoint eigenfunctions $\bar{\Phi}(\zeta')$ at a different eigenvalue, we derive

$$\int (\Phi_2, \Phi_5)(\zeta)\,\partial\,\hat{g}^{-1}\begin{pmatrix}\Phi_2\\ \Phi_5\end{pmatrix}(\zeta')\,dx = 0 \tag{4.11}$$

when $\zeta \neq \zeta'$.

Because of equation (3.13) we may consider Φ_2 and Φ_5 as the functional derivatives of ζ with respect to u^2 and u^5 respectively. Therefore, equation (3.3) may be considered as expressing the involution of distinct eigenvalues with respect to the Poisson bracket

$$\{H, K\} = \int \left(\frac{\delta H}{\delta u^2}, \frac{\delta H}{\delta u^5} \right) \partial \hat{g}^{-1} \begin{pmatrix} \frac{\delta K}{\delta u^2} \\ \frac{\delta K}{\delta u^5} \end{pmatrix} dx \tag{4.12}$$

Hence

$$\partial_t \begin{pmatrix} u^2 \\ u^5 \end{pmatrix} = -\partial \hat{g}^{-1} \begin{pmatrix} \frac{\delta H}{\delta u^2} \\ \frac{\delta H}{\delta u^5} \end{pmatrix} \tag{4.13}$$

are the Hamiltonian equations. It may be checked that the Hamiltonians of distinct flows, generated by the recursion operator M^+, are in involution with respect to this Poisson bracket.

<u>Some Polynomial Flows</u>

The first two non-trivial flows (2.16) and (2.18) may both be written as

$$\partial_t \begin{pmatrix} u^2 \\ u^5 \end{pmatrix} = M^+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.14}$$

where the co-efficients of each are determined by the constants of integration in M^+. Higher flows may be generated by successive application of M^+; the subsequent constants of integration may then be set equal to zero as they contribute nothing new. However, the calculations involved are rather tedious, so will be omitted here. Nevertheless, it should be pointed out that the integrations involved in these calculations can always be performed, the integrands being derivatives of

the momentum density

$$H_1 = \tfrac{1}{3}(u^2)^2 + (u^5)^2 \tag{4.15}$$

or the modified Boussinesq Hamiltonian density

$$H_2 = u^5 u^2_x - u^2 u^5_x + 2u^5\left((u^5)^2 - (u^2)^2\right) \tag{4.16}$$

'along' an isospectral flow, and are hence exact space derivatives, since these densities are conserved.

A special case of some interest is the restriction

$$u^5 \equiv 0 \tag{4.17}.$$

This corresponds to choosing the scalar operator L of (2.4) to be skew-Hermitian (the choices $u^2 \pm u^5 = 0$ are easily seen to be equivalent). With this restriction the recursion operator M^+ becomes

$$M^+\Big|_{u^5 \equiv 0} = \tfrac{1}{3}\partial \begin{pmatrix} 0 & (u^2+\partial)^2 - 2u^2\partial^{-1}u^2(\partial + u^2) \\ -\frac{1}{3}(u^2-\partial)^2 + 2\left((u^2)^2 - \partial u^2\right)\partial^{-1}u^2 & 0 \end{pmatrix} \tag{4.18}.$$

It is now apparent that $M^+\big|_{u^5\equiv 0}$ must be applied twice in order that the condition (3.37) be retained. The simplest nontrivial flow which may be generated in this way is thus given by

$$\begin{aligned} \partial_t u^2 &= \tfrac{1}{3}\partial\left[(u^2+\partial)^2 - 2u^2\partial^{-1}u^2(\partial+u^2)\right]\tfrac{1}{3}\partial\left[-\tfrac{1}{3}(u^2-\partial)^2 + 2\left((u^2)^2 - \partial u^2\right)\partial^{-1}u^2\right].0 \\ &= \tfrac{1}{9}\partial\left[(u^2+\partial)^2 - 2u^2\partial^{-1}u^2(\partial+u^2)\right]\partial\left[2\left((u^2)^2 - \partial u^2\right)\right]K_1 \\ &= \left[u^2_{4x} - 5u^2_x u^2_{2x} - 5u^2(u^2_x)^2 - 5(u^2)^2u^2_{2x} + (u^2)^5\right]_x \end{aligned} \tag{4.19}$$

on giving the constant of integration, K_1, the value $-\frac{9}{2}$. This is the system referred to in I as the 'modified Kupershmidt equation'.

5. NONLINEAR KLEIN-GORDON EQUATIONS

In order to generate isospectral flows of nonlinear Klein-Gordon type we require solutions of (2.14) for which A is inversely proportional to ζ :

$$A^k = \frac{a^k}{\zeta} \tag{5.1}$$

We write u^k in potential form

$$u^k = \theta^k{}_x \tag{5.2}$$

Equations (2.14) may be simplified considerably; to see this, we consider the third member of this set:

$$\partial a^3 = \zeta a^2 - \theta^2{}_x(a^1 + a^3) - \theta^5{}_x a^6 \tag{5.3}$$

Since the a's are independent of ζ, a^2 must vanish. By similar arguments we get

$$a^1 = a^2 = a^3 = a^5 = a^6 = 0 \tag{5.4}$$

Equations (2.14) then reduce to

$$\begin{aligned} a^4{}_x &= 3\theta^5{}_x(a^7 - a^8) - \theta^2{}_x a^4 \\ a^7{}_x &= \theta^2{}_x(a^7 + a^8) + \theta^5{}_x a^4 \\ a^8{}_x &= 2(\theta^2{}_x a^7 - \theta^5{}_x a^4) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \theta^2{}_{xt} &= 3a^7 \\ \theta^5{}_{xt} &= -a^4 \end{aligned} \tag{5.6}$$

To solve (5.5) we require that the a^k should be functions of θ^2 and θ^5 alone.

We thus obtain

$$\begin{aligned} a^4 &= -\tfrac{3}{2}\beta \exp(3\theta^5 - \theta^2) + \tfrac{3}{2}\gamma \exp(-\theta^2 - 3\theta^5) \\ a^7 &= \alpha \exp(2\theta^2) - \tfrac{1}{2}\beta \exp(3\theta^5 - \theta^2) - \tfrac{1}{2}\gamma \exp(-\theta^2 - 3\theta^5) \\ a^8 &= \alpha \exp(2\theta^2) + \beta \exp(3\theta^5 - \theta^2) + \gamma \exp(-\theta^2 - 3\theta^5) \end{aligned} \tag{5.7}$$

The equations of motion (5.6) then become

$$\begin{aligned} \theta^2{}_{xt} &= 3\alpha \exp(2\theta^2) - \tfrac{3}{2}\beta \exp(3\theta^5 - \theta^2) - \tfrac{3}{2}\gamma \exp(-\theta^2 - 3\theta^5) \\ \theta^5{}_{xt} &= \tfrac{3}{2}\beta \exp(3\theta^5 - \theta^2) - \tfrac{3}{2}\gamma \exp(-\theta^2 - 3\theta^5) \end{aligned} \tag{5.8}$$

If α, β and γ are all strictly positive, then without loss of generality they may all be taken to be $\frac{1}{3}$. Hence

$$\begin{aligned} \theta^2{}_{xt} &= \exp(2\theta^2) - \exp(-\theta^2)\cosh(3\theta^5) \\ \theta^5{}_{xt} &= \exp(-\theta^2)\sinh(3\theta^5) \end{aligned} \tag{5.9}$$

If θ^5 is allowed to take imaginary values then the system (5.9) possesses multi-soliton solutions.

These systems and some generalisations of them related to the Toda lattice have been discussed in more detail elsewhere[7], in which we show they possess a Bäcklund transformation. These general systems have also been considered by

Mikhailov[12] and Kupershmidt[13].

An interesting special case of (5.9) is when

$$\Theta^{5} \equiv 0 \tag{5.10}$$

which corresponds to the restriction (4.17). Then

$$\Theta^{2}_{\ 2t} = \exp(2\Theta^{2}) - \exp(-\Theta^{2}) \tag{5.11}.$$

This equation has been discussed by Dodd and Bullough[14] and Ibragimov[15].

6. CONCLUSIONS

This paper has had two main objects. The first of these was to extend the results of I to the general operator (1.1). The second was to develop a general scheme for constructing appropriate sets of squared eigenfunctions together with a recursion operator for an arbitrary scattering problem. We also constructed, for the eigenvalue problem (2.8), a Hamiltonian structure,as well as several iso-spectral flows.

The Miura transformation (2.6) which relates the Boussinesq system (2.2) to the modified system (2.18) may also be used to relate their Hamiltonian structures, in the manner described in I. However, starting from (4.12) we do not generate the usual Poisson bracket for the Boussinesq equation, but the 'second' Hamiltonian structure discussed by Kupershmidt and Wilson[16].

The methods of sections 3 and 4 are applicable to any simple Lie algebra; for $sl(2,\mathbb{R})$ the usual recursion operator and squared eigenfunction equations discussed by Ablowitz et al. [5] are generated. In the case of the Boussinesq equation this relationship between the adjoint representation and the squared eigenfunctions has been noted by Flaschka[17].

The results of section 5 can be generalised, not only to $sl(n,\mathbb{R})$, giving Klein-Gordon equations related to the Toda lattice, but also to any simple Lie algebra, giving systems related to Bogoyavlensky's[18] generalised exponential lattices[19].

APPENDIX:

Throughout this paper the following basis is used for $sl(3,\mathbb{R})$:

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 0 \end{pmatrix} \qquad R_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_7 = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad R_8 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The commutation relations for these are given by:

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1		R_3	$3R_4$	R_5	$-3R_6$	$-R_7$	$-3R_2$	0
R_2			$-2R_1-R_3$	$-R_4$	0	R_6	R_7+2R_8	R_7
R_3				$-R_5$	$3R_6$	$2R_8$	$-3R_2$	$-3R_2$
R_4					$2R_8-R_7$	$-R_2$	$2R_1$	$-R_3$
R_5						$2R_1-R_3$	$3R_4$	$-3R_4$
R_6							$-R_5$	R_5
R_7								$3R_6$
R_8								

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